MATH4210: Financial Mathematics Tutorial 3

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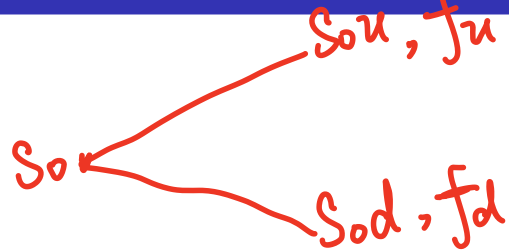
Binomial Tree Models



$$f = x = (1+r\Delta t)^{-1}[q f_u + (1-q) f_d]$$

$$\phi = \frac{f_u - f_d}{S_0u - S_0d}$$

$$q = \frac{1+r\Delta t - d}{u-d}$$

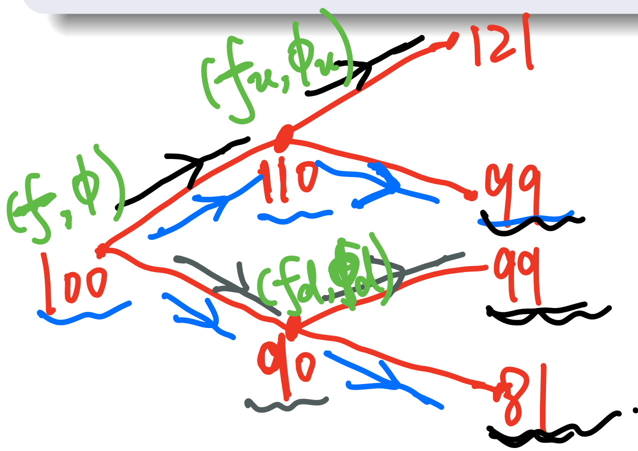
Question

~~max~~

A ***Lookback call*** *is idential to a stanard European call, except that the strike price is not set in advance, but is equal to the minimum price experienced by the underlying asset during the life of the call. Suppose the stock price* $S_0 = 100$, $u = 1.1$, $d = 0.9$ *in each of the next two years, and one step interest* $r = 0.02$. *What is the price and the replication strategy of a two-year Lookback call option?*

$1+r\Delta t = 1.02$; $q = \frac{1+r\Delta t - d}{u-d} = \frac{1.02-0.9}{1.1-0.9} = 0.6$

payoff $(S_T - K)^+$



$(121 - 100)^+ = 21 = f_{uu}$ $\quad f_u = 1.02^{-1}[0.6\times 21 + 0.4\times 0] = 12.35$

$(99 - 99)^+ = 0 = f_{ud}$ $\quad \phi_u = \frac{21-0}{121-99} = \frac{21}{22}$

$(99 - 90)^+ = 9 = f_{du}$ $\quad f_d = 1.02^{-1}[0.6\times 9 + 0.4\times 0] = 5.29$

$(81 - 81)^+ = 0 = f_{dd}$ $\quad \phi_d = \frac{9-0}{99-81} = \frac{1}{2}$

$$\text{price} = f = 1.02^{-1}[0.6\times 12.35 + 0.4\times 5.29]$$
$$= 9.34$$

$$\phi = \frac{12.35 - 5.29}{110 - 90} = 0.35$$

$t_0 =$ buy 0.35 shares of stock.

$t_1 =$

- buy extra $\frac{21}{22} - 0.35$ shares to hold $\frac{21}{22}$ shares.
- buy extra $\frac{1}{2} - 0.35$ shares to hold $\frac{1}{2}$ shares.

Binomial Tree Models

Question

Suppose you are given a two step binomial tree model with the following: $S_{t_0} = 100$, $u = 1.05$, $d = 0.95$, $r = 0.02$. *Consider a two period* ***Asian call option*** *where the averaging is done over all three prices observed.*

$(S_T - K)^+$

(1) Suppose the option is an ***averaging − price*** *Asian option with a strike price of* 100 *and only can be exercised at time* t_2. *Find the initial price and the replication strategy.*

(2) Suppose the option is an ***averaging − strike*** *Asian option and only can be exercised at time* t_2. *Find the initial price and the replication strategy.*

average of three prices observed

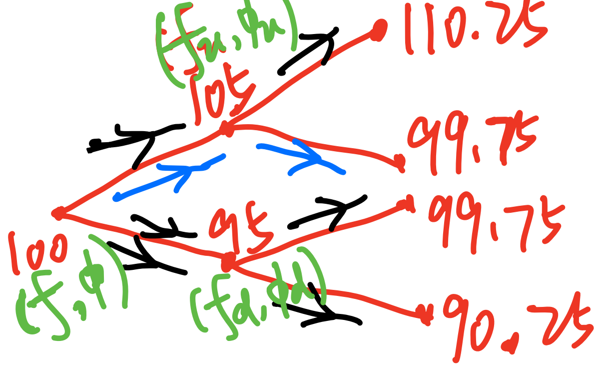
$1+r\Delta t = 1.02$

$q = \frac{1+r\Delta t - d}{u-d} = 0.7$

payoff $(\bar{S}_T - 100)^+$

(1)

$f_u = 1.02^{-1}[0.7 \times 5.08 + 0.3 \times 1.58]$



$=3.95$

$\left(\frac{100+105+110.25}{3}-100\right)^+=5.08$

$\left(\frac{100+105+99.75}{3}-100\right)^+=1.58$

$\left(\frac{100+95+99.75}{3}-100\right)^+=0$

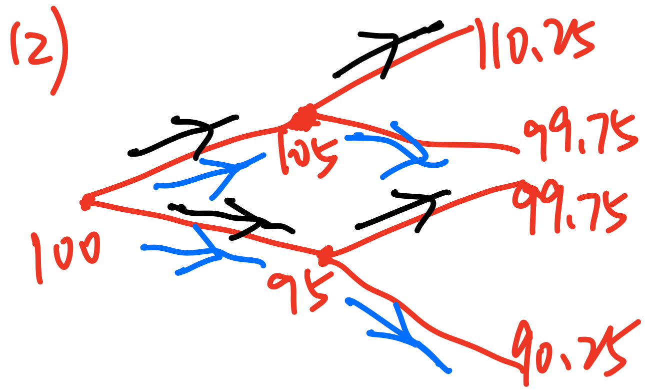
$\left(\frac{100+95+90.25}{3}-100\right)^+=0$

$\phi_u=\frac{5.08-1.58}{110.25-99.75}=0.33$

$f_d=1.02^{-1}[0.7\times 0+0.3\times 0]=0$

$\phi_d=\frac{0-0}{99.75-90.25}=0$

(2)



payoff $(S_T-K)^+$

$\left(110.25-\frac{100+105+110.25}{3}\right)^+=5.17$

$\left(99.75-\frac{100+105+99.75}{3}\right)^+=0$

$\left(99.75-\frac{100+95+99.75}{3}\right)^+=1.5$

$\left(90.25-\frac{100+95+90.25}{3}\right)^+=0$

price $=f=1.02^{-1}[0.7\times 3.95+0.3\times 0]$

$=2.71$

$\phi=\frac{3.95-0}{105-95}=0.395$

t_0: buy 0.395 shares.

t_1: sell 0.395−0.33 shares to hold 0.33 shares.

sell all 0.395 shares.

Binomial Tree Models

q u

$1-q$ d

Question

Consider a sequence of i.i.d. random variables $\{\xi_k\}_{k\in\mathbb{N}^*}$ which takes value u with probability q and d with probability $1-q$ with q the risk neutral probability. Then the stock price can be written as $S_n = S_0\Pi_{k=1}^n \xi_k$. Show that the discounted stock price is a discrete martingale.

$\Delta t=1$

$X_n=(1+r)^{-n}S_n$

Answer

First, we need to prove the integrability. Fix $n > 0$.

x variable

independent

$$
\begin{aligned}
(1)\quad \mathbb{E}^{\mathbb{Q}}[|(1+r)^{-n}S_n|] &= (1+r)^{-n}\mathbb{E}^{\mathbb{Q}}[S_0\Pi_{k=1}^n\xi_k] \\
&= (1+r)^{-n}S_0\Pi_{k=1}^n\mathbb{E}^{\mathbb{Q}}[\xi_k] \quad qu+(1-q)d \\
&= (1+r)^{-n}S_0(qu+d-qd)^n \\
&< \infty
\end{aligned}
$$

Binomial Tree Models

Answer

(2) $\mathcal{F}_n = \sigma(S_1, \dots, S_n)$ ✓ $\Rightarrow X_n \in \mathcal{F}_n$

$\in \mathcal{F}_n$

$E(S_n \mid \mathcal{F}_n) = S_n$

$E(\xi_{n+1} \mid \mathcal{F}_n) = E(\xi_{n+1})$

Second, for $n > 0$,

(3)

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[(1+r)^{-(n+1)} S_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}^{\mathbb{Q}}[(1+r)^{-n} S_n * (1+r)^{-1} \xi_{n+1} \mid \mathcal{F}_n] \\
&= (1+r)^{-n} S_n * (1+r)^{-1} \mathbb{E}^{\mathbb{Q}}[\xi_{n+1} \mid \mathcal{F}_n] \\
&= (1+r)^{-n} S_n * (1+r)^{-1} \mathbb{E}^{\mathbb{Q}}[\xi_{n+1}] \\
&= (1+r)^{-n} S_n * \frac{qu + (1-q)d}{1+r} \\
&= (1+r)^{-n} S_n * \frac{(1+r)(u-d)}{(1+r)(u-d)} = 1 \\
&= (1+r)^{-n} S_n \\
&= X_n
\end{aligned}
$$

$E^{Q}[X_{n+1} \mid \mathcal{F}_n] = X_n$ for all n

$n+1$

$\in \mathcal{F}_n$

$S_0 \prod_{i=1}^{n} \xi_i$

$\sigma(S_1, \dots, S_n)$

Hence, the discounted stock price is a discrete martingale under probability measure $\mathbb{Q}$.